

Quantum Linear Coherent Controller Synthesis: A Linear Fractional Representation Approach[★]

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Abstract

This paper is concerned with a linear fractional representation approach to the synthesis of linear coherent quantum controllers for a given linear quantum plant. The plant and controller represent open quantum harmonic oscillators and are modelled by linear quantum stochastic differential equations. The feedback interconnections between the plant and the controller are assumed to be established through quantum bosonic fields. In this framework, conditions for the stabilization of a given linear quantum plant via linear coherent quantum feedback are addressed using a stable factorization approach. The class of all stabilizing quantum controllers is parameterized in the frequency domain. Coherent quantum weighted \mathcal{H}_2 and \mathcal{H}_∞ control problems for linear quantum systems are formulated in the frequency domain. Finally, a projected gradient descent scheme is outlined for the coherent quantum weighted \mathcal{H}_2 control problem.

Key words: Coherent quantum control, linear quantum stochastic systems, linear fractional representation, frequency domain.

1 Introduction

The main motivation for coherent quantum feedback control is based on avoiding the loss of quantum information in conversion to classical signals which occurs during measurement [23][20]. This approach builds on the technique of constructing a feedback network from the interconnection of quantum systems, for example, through field coupling; see [12,13]. In this framework, coherent quantum control theory aims at developing systematic methods to design measurement-free interconnections of Markovian quantum systems modelled by quantum stochastic differential equations (QSDEs); for example, see [17,25,28]. Owing to recent advances in quantum optics, the implementation of quantum feedback networks governed by linear QSDEs [27,21,28] is possible using quantum-optical components, such as optical cavities, beam splitters and phase shifters, provided the former represent open quantum harmonic oscillators (OQHOs) with a quadratic Hamiltonian and linear system-field coupling operators with respect to the state variables satisfying canonical commutation relations [8,10]. This important class of linear QSDEs models the Heisenberg evolution of pairs of conjugate operators in a multi-mode quantum harmonic oscillator that is coupled to external bosonic fields. As a consequence, the notion of physical realizability (PR) addresses conditions under which a linear QSDE represents an OQHO. This condition is organised as a set of constraints on the coefficients of the QSDE [17] or, alternatively, on the quantum system transfer matrix [31,32] in the frequency domain. These constraints complicate the solution of the coherent quantum synthesis problems which are otherwise reducible to tractable unconstrained counterparts in classical control theory.

Coherent quantum feedback control problems, such as internal stabilization and optimal control design, are of particular interest in linear quantum control theory [17,28]. These problems are amenable to transfer matrix design methods [36,37,13,28,31]. Among the transfer matrix approaches to the control problems for linear multivariable systems, the linear fractional representation approach to analysis and synthesis has been largely developed in the literature; see [34] and the references therein. The linear fractional representation approach is a cornerstone in the study of stabilization problems. The central idea of this approach is to represent the transfer matrix of a plant as fractions of stable rational matrices to generate stable factorizations. By

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combining the idea of the stable factorizations of a plant with the concept of coprimeness, necessary and sufficient conditions for internal stabilizability are derived in terms of Bézout equations [34]. By solving these Bézout equalities, a parameterization of all stabilizing controllers, known as the Youla-Kučera parameterization, is obtained. This idea gives rise to a method which leads to the solution of several important control problems; see for example [34].

The Youla-Kučera parameterization was developed originally in the frequency domain for finite-dimensional linear time-invariant systems using transfer function methods, see [38,39], and generalized to infinite-dimensional systems afterwards [7,29,34]. The state space representation of all stabilizing controllers has also been addressed for finite-dimensional, linear time-invariant [24] and time-varying [6] systems, and the approach was shown to be applicable to a class of nonlinear systems [14,26,1]. In the Youla-Kučera parameterization, the feedback loop involving the controller is redefined in terms of another parameter known as the Youla or Q parameter. The closed-loop map is then an affine function of Q , and so the optimal Q in standard optimal stabilization problems can be easily found. Moreover, some constraints, such as internal stability, are reduced to convex constraints on Q . Therefore, this approach provides a tool that allows us to better understand the dichotomy between tractable and intractable control synthesis problems in the presence of additional constraints on the controller; see for example [5].

In the present paper, we employ a stable factorization approach in order to develop a counterpart of the classical Youla-Kučera parameterization for describing the set of linear coherent quantum controllers that stabilize a linear quantum stochastic system (LQSS). In particular, we address the problem of coherent quantum stabilizability of a given linear quantum plant. The class of all stabilizing controllers is parameterized in the frequency domain. This approach allows weighted \mathcal{H}_2 and \mathcal{H}_∞ coherent quantum control problems to be formulated for linear quantum systems in the frequency domain. In this way, the weighted \mathcal{H}_2 and \mathcal{H}_∞ control problems are reduced to constrained optimization problems with respect to the Youla-Kučera parameter with convex cost functionals. Moreover, these problems are organised as a constrained version of the model matching problem [9]. Finally, a projected gradient descent scheme is proposed to provide a conceptual solution to the weighted \mathcal{H}_2 coherent quantum control problem in the frequency domain.

The rest of this paper is organised as follows. Section 2 outlines the notation used in the paper. Linear quantum stochastic systems are described in Section 3. The coherent quantum feedback interconnection under consideration is described in Section 4. Section 5 revisits the PR conditions for linear quantum systems in the frequency domain. Sections 6 and 7 formulate a quantum version of the Youla-Kučera parameterization and provide relevant preparatory material. Also, a class of unstabilizable LQSS systems is presented. Coherent quantum weighted \mathcal{H}_2 and \mathcal{H}_∞ control problems are introduced in Section 8. A projected gradient descent scheme for the quantum weighted \mathcal{H}_2 control problem is outlined in Section 9. Section 10 gives concluding remarks. Appendix A provides a parameterization of linear coherent quantum feedback systems in the position-momentum form. Appendix B provides a Cholesky-like factorization for skew-symmetric matrices. Appendices C and D provide relevant facts about linear fractional transformations and the general Bézout identity. Appendices E and F provide complementary materials for purposes of Section 7 and Section 9.

A preliminary version of this work has been published in the conference proceedings of the 10th Asian control conference in 2015 [33]. In comparison to the conference version, use is made of a modified version of the physical realizability condition for linear quantum stochastic systems in the frequency domain [32] which leads to more complete and simple results. The changes include a real-valued parameterization of the linear coherent quantum stochastic feedback systems (without loss of generality) and the omission of technical assumptions in the main results of the paper. The main theorem, Theorem 8, in [33] and its proof has been modified to provide a parameterization of the set of *all* stabilizing linear coherent quantum controllers. A class of linear quantum systems is presented which cannot be stabilized by linear coherent quantum controllers. Complementary results and technical details are presented in the appendices.

2 Notation

Unless specified otherwise, vectors are organized as columns, and the transpose $(\cdot)^T$ acts on matrices with operator-valued entries as if the latter were scalars. For a vector X of self-adjoint operators X_1, \dots, X_r and a vector Y of operators Y_1, \dots, Y_s , the commutator matrix is defined as an $(r \times s)$ -matrix $[X, Y^T] := XY^T - (YX^T)^T$ whose (j, k) th entry is the commutator $[X_j, Y_k] := X_j Y_k - Y_k X_j$ of the operators X_j and Y_k . Furthermore, $(\cdot)^\dagger := ((\cdot)^\#)^T$ denotes the transpose of the entry-wise operator adjoint $(\cdot)^\#$. When it is applied to complex matrices, $(\cdot)^\dagger$ reduces to the complex conjugate transpose $(\cdot)^* := (\overline{(\cdot)})^T$. The positive semi-definiteness of matrices is denoted by \succ , and \otimes is the tensor product of spaces or operators (for example, the Kronecker product of matrices). Furthermore, \mathbb{S}_r , \mathbb{A}_r and $\mathbb{H}_r := \mathbb{S}_r + i\mathbb{A}_r$ denote the subspaces of real symmetric, real antisymmetric and complex Hermitian matrices of order r , respectively, with $i := \sqrt{-1}$ the imaginary unit. Also, I_r denotes the identity matrix

of order r , the identity operator on a space \mathcal{H} is denoted by $\mathcal{I}_{\mathcal{H}}$, and the matrices $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $J_r := I_{\frac{r}{2}} \otimes J$. The sets $O(r) := \{\Sigma \in \mathbb{R}^{r \times r} : \Sigma^T \Sigma = I\}$ and $\text{Sp}(r, \mathbb{R}) := \{\Sigma \in \mathbb{R}^{r \times r} : \Sigma^T J_r \Sigma = J_r\}$ refer to the group of orthogonal matrices and the group of symplectic real matrices of order r . The notation $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ refers to a state-space realization of the corresponding transfer matrix $\Gamma(s) := C(sI - A)^{-1}B + D$ with a complex variable $s \in \mathbb{C}$. The conjugate system transfer function $(\Gamma(-\bar{s}))^*$ is written as $\Gamma^\sim(s)$. The Hardy space of (rational) transfer functions of type $p = 2, \infty$ is denoted by \mathcal{H}_p (respectively, \mathcal{RH}_p). The symbol \otimes is used for the tensor product of spaces.

3 Linear Quantum Stochastic Systems

We consider a Markovian quantum stochastic system interacting with an external boson field. The system has n dynamic variables $X_1(t), \dots, X_n(t)$, where $t \geq 0$ denotes time. We generally suppress the time argument of operators, unless we are explicitly concerned with their time dependence, with the understanding that all operators are evaluated at the same time. The system variables are self-adjoint operators on an underlying complex separable Hilbert space \mathcal{H} which satisfy the Heisenberg canonical commutation relations (CCRs)

$$[X, X^T] = 2i\Theta \otimes \mathcal{I}_{\mathcal{H}}, \quad X := \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}, \quad (1)$$

on a dense domain in \mathcal{H} , where $\Theta \in \mathbb{A}_n$ is nonsingular. In what follows, the matrix $\Theta \otimes \mathcal{I}_{\mathcal{H}}$ will be identified with Θ . The system variables evolve in time according to a Hudson-Parthasarathy QSDE [27] with identity scattering matrix (which eliminates from consideration the gauge, also known as conservation, processes associated with photon exchange between the fields):

$$dX = fdt + gdw. \quad (2)$$

The n -dimensional drift vector f and the dispersion $(n \times m)$ -matrix g of the QSDE (2) are given by

$$f := \mathcal{L}(X) = i[H, X] + \mathcal{D}(X), \quad g := -i[X, L^T], \quad L := \begin{bmatrix} L_1 \\ \vdots \\ L_m \end{bmatrix}. \quad (3)$$

Here, H is the system Hamiltonian which is usually represented as a function of the system variables, and L_1, \dots, L_m are the system-field coupling operators (the dimension m is assumed to be even). These self-adjoint operators act on the space \mathcal{H} and specify the self-energy of the system and its interaction with the environment. Furthermore, \mathcal{L} coincides with the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) generator [11, 19] which acts on a system operator ξ as

$$\mathcal{L}(\xi) := i[H, \xi] + \mathcal{D}(\xi) \quad (4)$$

and is evaluated entry-wise at the vector X in (1), and \mathcal{D} is the decoherence superoperator given by

$$\begin{aligned} \mathcal{D}(\xi) &:= \frac{1}{2} \sum_{j,k=1}^m \omega_{jk} ([L_j, \xi] L_k + L_j [\xi, L_k]) \\ &= \frac{1}{2} ([L^T, \xi] \Omega L + L^T \Omega [\xi, L]). \end{aligned} \quad (5)$$

In the QSDE (2), W is an m -dimensional vector of quantum Wiener processes W_1, \dots, W_m , which are self-adjoint operators on a boson Fock space [18, 27], modelling the external fields with a positive semi-definite Itô matrix $\Omega := (\omega_{jk})_{1 \leq j, k \leq m} \in \mathbb{H}_m$:

$$dW dW^T = \Omega dt. \quad (6)$$

The entries of W are linear combinations of the field annihilation $\mathfrak{A}_1, \dots, \mathfrak{A}_{\frac{m}{2}}$ and creation $\mathfrak{A}_1^\dagger, \dots, \mathfrak{A}_{\frac{m}{2}}^\dagger$ operator processes [16,27]:

$$W := 2P_m \begin{bmatrix} \text{Re}\mathfrak{A} \\ \text{Im}\mathfrak{A} \end{bmatrix} = P_m T_m \begin{bmatrix} \mathfrak{A} \\ \mathfrak{A}^\# \end{bmatrix}, \quad T_m := \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \otimes I_{\frac{m}{2}}, \quad (7)$$

where

$$\mathfrak{A} := \begin{bmatrix} \mathfrak{A}_1 \\ \vdots \\ \mathfrak{A}_{\frac{m}{2}} \end{bmatrix}, \quad \mathfrak{A}^\# := \begin{bmatrix} \mathfrak{A}_1^\dagger \\ \vdots \\ \mathfrak{A}_{\frac{m}{2}}^\dagger \end{bmatrix},$$

with the quantum Itô relations

$$d \begin{bmatrix} \mathfrak{A} \\ \mathfrak{A}^\# \end{bmatrix} d \begin{bmatrix} \mathfrak{A}^\dagger & \mathfrak{A}^T \end{bmatrix} := \begin{bmatrix} d\mathfrak{A}d\mathfrak{A}^\dagger & d\mathfrak{A}d\mathfrak{A}^T \\ d\mathfrak{A}^\#d\mathfrak{A}^\dagger & d\mathfrak{A}^\#d\mathfrak{A}^T \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{\frac{m}{2}} \right) dt,$$

where $P_m \in \mathbb{R}^{m \times m}$ is a permutation matrix such that $P_m(J \otimes I_{\frac{m}{2}})P_m^T = I_{\frac{m}{2}} \otimes J = J_m$. Accordingly, the Itô matrix Ω in (6) is described by

$$\Omega = P_m T_m \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{\frac{m}{2}} \right) T_m^* P_m^T = I_m + iJ_m = \Omega^* \succcurlyeq 0. \quad (8)$$

In accordance with the evolution (2), the system variables $X_1(t), \dots, X_n(t)$ at any given time $t \geq 0$ act on a tensor product Hilbert space $\mathcal{H}_0 \otimes \mathcal{F}_t$, where \mathcal{H}_0 is the initial complex separable Hilbert space of the system and \mathcal{F}_t is the Fock filtration. By using the quantum stochastic calculus [27], in accordance with (3)–(5), the QSDE (2) is derived from the Heisenberg unitary evolution on the tensor product of system and field spaces $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{F}_t$ described by the quantum stochastic flow

$$X(t) = U(t)^\dagger (X(0) \otimes \mathcal{I}_{\mathcal{F}}) U(t), \quad (9)$$

where the unitary operator $U(t)$ satisfies the initial condition $U(0) = \mathcal{I}_{\mathcal{H}}$ and is governed by a stochastic Schrödinger equation

$$dU(t) = - \left((iH(0) + \frac{1}{2}L(0)^T \Omega L(0)) dt + iL(0)^T dW(t) \right) U(t). \quad (10)$$

The unitary evolution in (10) preserves the CCRs (1):

$$\begin{aligned} [X(t), X(t)^T] &= U(t)^\dagger ([X(0), X(0)^T] \otimes \mathcal{I}_{\mathcal{F}}) U(t) \\ &= 2i\Theta U(t)^\dagger \mathcal{I}_{\mathcal{H}_0 \otimes \mathcal{F}} U(t) = 2i\Theta, \end{aligned}$$

where the entries of $X(0)$ commute with those of $W(t)$ as operators on different spaces.

In particular, by considering the following Hamiltonian and system-field coupling operators

$$H = \frac{1}{2} X^T R X = \frac{1}{2} \sum_{j,k=1}^n r_{jk} X_j X_k, \quad L = M X, \quad (11)$$

the system corresponds to an $\frac{n}{2}$ -mode OQHO [8,10]. Here, $R := (r_{jk})_{1 \leq j,k \leq n}$ is a real symmetric matrix of order n , and $M \in \mathbb{R}^{m \times n}$ is the system-field coupling parameter. By substituting the Hamiltonian and coupling operators from (11) into (3) and using the CCRs (1), it follows that the QSDE takes the form

$$dX(t) = AX(t)dt + BdW(t), \quad (12)$$

where, in view of (2), the drift vector $f = AX$ and the dispersion matrix $g = B$ are given by

$$A := 2\Theta R - \frac{1}{2} B J_m B^T \Theta^{-1}, \quad B := 2\Theta M^T. \quad (13)$$

The term $-\frac{1}{2}BJ_mB^T\Theta^{-1}X$ in the drift represents the GKSL decoherence superoperator which acts on the system variables and is associated with the system-field interaction.

We associate a vector Y of output field dynamic variables Y_1, \dots, Y_p with an OQHO:

$$dY = CXdt + DdW, \quad (14)$$

where $C \in \mathbb{R}^{p \times n}$ with the dimension p assumed to be even and the full row rank matrix $D \in \mathbb{R}^{p \times m}$ satisfies

$$D(I_m + iJ_m)D^T = I_p + iJ_p \quad (15)$$

from a similar condition to (6) for the outputs ($p \leq m$). The output field satisfies the non-demolition condition [3] with respect to the dynamic variables in the sense that

$$[X(t), Y(s)^T] = 0, \quad t \geq s. \quad (16)$$

Due to this non-demolition property, the output fields can be interpreted as ideal observations of the open quantum system, except that Y_1, \dots, Y_p do not commute with each other. This condition implies an algebraic relation between C and D in (14):

$$C = -DJ_mB^T\Theta^{-1}. \quad (17)$$

We will refer to the input-output dynamics of a system S which is described by (12) and (14) as a LQSS. This system can be parameterized by the triple (D, M, R) in (13) and (17). By analogy with classical linear systems, we often represent this system by the state-space realization $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. The input-output block-diagram of the LQSS is depicted as in Fig. 1. Note that there

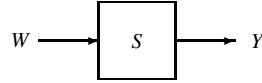


Fig. 1. Input-output block-diagram.

exists a one-to-one correspondence between the real-valued (D, M, R) parameterization of the LQSS, which will be referred to as the *position-momentum form*, and the complex-valued, but structured, parameterization, referred to as *annihilation-creation form* [32].

4 Coherent Quantum Feedback Interconnection

In this section, we will provide a framework for the interconnection of two LQSSs, one acting as the quantum plant and the other as the controller. In this framework, we consider field interconnections between the two systems with initial complex separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. In particular, the vectors of dynamic variables of the plant and the controller are denoted by X_1 and X_2 (which consist of self-adjoint operators on the product space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{F}_1 \otimes \mathcal{F}_2$ at any subsequent moment of time $t > 0$) and are assumed to satisfy CCRs

$$[X, X^T] = 2i\Theta, \quad X := \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \Theta := \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}, \quad (18)$$

where $\Theta_1, \Theta_2 \in \mathbb{A}_n$ are constant nonsingular matrices.

By analogy with similar structures in the interconnections arising in classical control theory, we partition the vectors W and Y of the plant P input and output field operators in accordance with Fig. 2:

$$W = \begin{bmatrix} W_r \\ W_u \end{bmatrix}, \quad Y = \begin{bmatrix} Y_z \\ Y_y \end{bmatrix}. \quad (19)$$

Here W_r, Y_z, W_u, Y_y denote the vectors of conjugate pairs of the input and output fields of the closed-loop system, and the input and output of K , which correspond to the classical reference, output, control and observation signals, respectively. The field

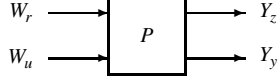


Fig. 2. This diagram depicts the way in which the input and output field conjugate pairs of the system P are partitioned in (19). This structure allows for field coupling to another quantum system.

coupling feedback interconnection of the systems P and K is shown in Fig. 3. Note that, similarly to the classical case, the interconnection in Fig. 3 provides a general framework for the feedback interconnection of two quantum systems. Note that

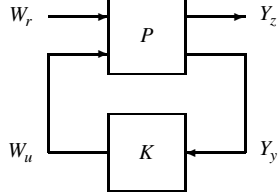


Fig. 3. This diagram depicts the fully quantum closed-loop system which is the interconnection of the quantum systems P and K . The effect of the environment on the closed-loop system is represented by W_r .

in the case of quantum control, both the plant and controller will have exogenous inputs and outputs. However, this situation can be handled in the framework of Figure 3. Indeed, this framework includes the conventional coherent quantum feedback interconnection shown in Fig. 4, where P_1 and K act as the quantum plant and the controller, respectively. In this figure, the exogenous inputs and outputs of the closed-loop system are grouped together. Here, r_1 and r_2 represent the exogenous inputs of the plant and controller, respectively. Also, z_1 and z_2 represent the exogenous outputs of the controller and the plant, respectively.

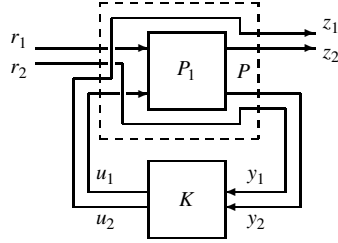


Fig. 4. This diagram depicts the way in which the quantum system P , the concatenation of P_1 and the feedthroughs, is formed by grouping the exogenous inputs and outputs of the closed-loop system.

Note that the feedback system in Fig. 3 represents a LQSS; see Lemma 13 in Appendix A.

5 Open Quantum Harmonic Oscillators in the Frequency Domain and Physical Realizability

In accordance with Section 3, we consider the dynamics of the joint evolution of an $\frac{n}{2}$ -mode OQHO and the external bosonic fields in the Heisenberg picture, represented by the linear QSDEs (12) and (14). Here, in view of the (D, M, R) parameterization of LQSSs provided in Section 3, the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$ in (12) and (14) are given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} 2\Theta R - \frac{1}{2}BJ_m B^T \Theta^{-1} & B \\ -DJ_m B^T \Theta^{-1} & D \end{bmatrix}, \quad B := 2\Theta M^T, \quad (20)$$

where $\Theta \in \mathbb{A}_n$ is the CCR matrix. Also, the parameter R is a real symmetric matrix of order n associated with the quadratic Hamiltonian of the OQHO, $M \in \mathbb{R}^{m \times n}$ is the system-field coupling parameter, the feedthrough real matrix D belongs to the subgroup of orthogonal symplectic matrices (the maximum compact subgroup of symplectic matrices)

$$\text{Sp}(m) = \text{O}(m) \cap \text{Sp}(m, \mathbb{R}). \quad (21)$$

The input-output map of the LQSS, governed by the linear QSDEs (12) and (14), is completely specified by a transfer function which is defined in the standard way as

$$\Gamma(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = C(sI - A)^{-1}B + D, \quad (22)$$

where the matrices A, B, C, D are parameterized by the triple (D, M, R) as in (20) with a given CCR matrix Θ . As discussed above, in view of the specific structure of this parameterization, not every linear system, or system transfer function (22) with an arbitrary quadruple (A, B, C, D) , represents the dynamics of a LQSS. This fact is addressed in the form of PR conditions for the quadruple (A, B, C, D) to represent such an oscillator; see [17] for more details. The notion of PR for a transfer function is defined as follows.

Definition 1 [32] *The transfer function $\Gamma(s)$ is said to be physically realizable if $\Gamma(s)$ represents a LQSS, that is, there exists a minimal state-space realization for $\Gamma(s)$ which can be parameterized by a triple (D, M, R) as in (20) for a given CCR matrix Θ .*

Note that, in view of the results of Lemma 14 in Appendix B, the invariance of transfer functions with respect to similarity transformations on the corresponding state-space realizations [40] and Definition 1, it can be shown that $\Gamma(s)$ is also physically realizable if there exists a minimal state-space realization for $\Gamma(s)$ which can be parameterized by the triple (D, M, R) as in (20) with any non-singular skew-symmetric matrix Θ . The following lemma provides a PR condition for transfer matrices of linear quantum systems, which can be considered as a modified version of Theorem 4 in [31]. The proof of this lemma is similar to the corresponding one in [32, Theorem 1] which is omitted for brevity. In what follows, the subscripts in I_m and J_m will often be omitted for brevity.

Lemma 2 *A transfer function $\Gamma(s)$ is physically realizable if and only if*

$$\Gamma^\sim(s)J\Gamma(s) = J \quad (23)$$

for all $s \in \mathbb{C}$, and the feedthrough matrix $D = \Gamma(\infty)$ is orthogonal.

A transfer function $\Gamma(s)$, satisfying the condition (23), is said to be (J, J) -unitary; see, for example, [31] and references therein. Since we consider this property for invertible square transfer matrices, in view of the fact that $J^2 = -I$, the (J, J) -unitary condition (23) is equivalent to its dual form [33]:

$$\Gamma(s)J\Gamma^\sim(s) = J. \quad (24)$$

In view of the one-to-one correspondence described in [32], the results in Lemma 2 imply the results in [31, Theorem 4]. However, in comparison to [31, Theorem 4], no additional technical assumptions are required in Lemma 2. The technical assumption which is used in [31] is referred to as spectral genericity of the linear quantum systems [33].

6 Parameterizations of All Stabilizing Controllers

We consider a linear quantum plant and a linear quantum controller with square transfer matrices P and K , respectively, each representing a LQSS in the frequency domain. In what follows, the argument s of transfer functions will often be omitted for brevity.

Following the field coupling feedback scheme introduced in Section 4, we assume that the plant and the controller are connected according to Fig. 3. To allow for the feedback interconnection, we partition the plant transfer matrix as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} =: \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]. \quad (25)$$

The closed-loop transfer matrix between the exogenous inputs and outputs of interest can be calculated through the lower linear fractional transformation (LFT) of the plant and the controller in the frequency domain [40, 13]:

$$G = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} =: \text{LFT}(P, K). \quad (26)$$

For the purposes of Section 7, we will now briefly review the classical Youla-Kučera parameterization of stabilizing controllers together with related notions. The latter include stabilizability, detectability, internal stability, coprime factorizations and matrix fractional descriptions (MFDs). Despite the quantum control context, these notions will be used according to their standard definitions in classical linear control theory [40,34].

6.1 Stabilizability of Feedback Connections

Consider the $(2, 2)$ block of the plant transfer matrix P in (25) given by

$$P_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array} \right]. \quad (27)$$

The following lemma provides a necessary and sufficient condition for the internal stability of the feedback system in Fig. 3.

Lemma 3 [40] *Suppose (A, B_2, C_2) in (27) is stabilizable and detectable. Then the closed-loop system in Fig. 3 is internally stable if and only if so is the system in Fig. 5.*

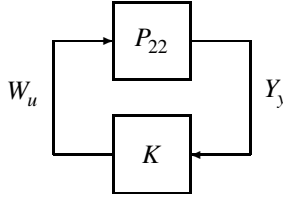


Fig. 5. Equivalent stabilization diagram.

6.2 Stable Factorization

Let the transfer function P_{22} in (27) have the following coprime factorizations over \mathcal{RH}_∞ :

$$P_{22} = NM^{-1} = \widehat{M}^{-1}\widehat{N}, \quad (28)$$

where the pairs (N, M) and $(\widehat{N}, \widehat{M})$ of transfer functions in \mathcal{RH}_∞ specify the right and left factorizations, respectively. Then there exist $U, V, \widehat{U}, \widehat{V} \in \mathcal{RH}_\infty$ which satisfy the Bézout identities:

$$\widehat{V}M - \widehat{U}N = I, \quad \widehat{M}V - \widehat{N}U = I. \quad (29)$$

The following lemma provides conditions for the existence of stable coprime factors for the system P_{22} .

Lemma 4 [40] *Suppose (A, B_2, C_2) in (27) is stabilizable and detectable. Then the coprime factorizations of P_{22} over \mathcal{RH}_∞ , described by (28), (29), can be realized by choosing*

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|c} A+B_2F & B_2-L \\ \hline F & I \quad 0 \\ C_2+D_{22}F & D_{22} \quad I \end{array} \right], \quad (30)$$

$$\begin{bmatrix} \widehat{V} & -\widehat{U} \\ -\widehat{N} & \widehat{M} \end{bmatrix} = \left[\begin{array}{c|c} A+LC_2 & -(B_2+LD_{22}) \quad L \\ \hline F & I \quad 0 \\ C_2 & -D_{22} \quad I \end{array} \right], \quad (31)$$

where $F \in \mathbb{R}^{\mu \times n}$ and $L \in \mathbb{R}^{n \times \mu}$ are such that both matrices $A+B_2F$ and $A+LC_2$ are Hurwitz. Furthermore, the systems in (30), (31) satisfy the general Bézout identity

$$\begin{bmatrix} \widehat{V} & -\widehat{U} \\ -\widehat{N} & \widehat{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (32)$$

6.3 The Youla-Kučera Parameterization

The main idea of the Youla-Kučera parameterization approach is built on stable factorizations (the representation of the transfer matrix of a plant as a fraction of stable rational matrices). By combining the idea of the stable factorizations of a plant with the concept of coprimeness, necessary and sufficient conditions for internal stabilizability are derived in terms of the general Bézout identities; see Appendix D. Solving the Bézout equations, the parameterization of all stabilizing controllers is obtained. The following lemma applies results on the Youla-Kučera parameterization in the frequency domain to the closed-loop system being considered; for more details, see [40] and the references therein.

Lemma 5 [40] *Suppose the block P_{22} of the plant transfer matrix P in (25) has the coprime factorizations over \mathcal{RH}_∞ , described by (28). Then the set of all controllers which achieve internal stability of the closed-loop system is parameterized either by*

$$K = (U + MQ_r)(V + NQ_r)^{-1}, \quad (33)$$

with $Q_r \in \mathcal{RH}_\infty$ satisfying

$$\det(V + NQ_r)(\infty) \neq 0, \quad (34)$$

or by

$$K = (\hat{V} + Q_\ell \hat{N})^{-1}(\hat{U} + Q_\ell \hat{M}), \quad (35)$$

with $Q_\ell \in \mathcal{RH}_\infty$ satisfying $\det(\hat{V} + Q_\ell \hat{N})(\infty) \neq 0$. Also, let the auxiliary transfer matrices $U, V, \hat{U}, \hat{V} \in \mathcal{RH}_\infty$ in (29) be chosen so that $UV^{-1} = \hat{V}^{-1}\hat{U}$, which is equivalent to (32); see Appendix D for more details. Then the set of all stabilizing controllers is parameterized by

$$\begin{aligned} K &= (U + MQ)(V + NQ)^{-1} \\ &= (\hat{V} + Q\hat{N})^{-1}(\hat{U} + Q\hat{M}) = \text{LFT}(O_y, Q), \end{aligned} \quad (36)$$

where the parameter $Q \in \mathcal{RH}_\infty$ of these factorizations satisfies

$$\det(V + NQ)(\infty) \neq 0, \quad (37)$$

and O_y is an auxiliary system given by

$$O_y := \begin{bmatrix} UV^{-1} & \hat{V}^{-1} \\ V^{-1} & -V^{-1}N \end{bmatrix}. \quad (38)$$

In what follows, the class of stabilizing controllers will be parameterized using MFDs. However, they can also be parameterized in the LFT framework due to the relationship between MFD and LFT representations; see Appendix C for more details.

7 Quantum Version of the Youla-Kučera Parameterization

We will now employ the results of Sections 5–6 in order to describe stabilizing coherent quantum controllers in the frequency domain. The following lemma represents the (J_μ, J_μ) -unitary condition in terms of the Youla-Kučera parameter Q from (36).

Lemma 6 *Suppose the controller transfer matrix K is factorized according to (36). Then K is (J_μ, J_μ) -unitary if and only if the parameter $Q \in \mathcal{RH}_\infty$ satisfies*

$$\Phi + Q^\sim \Lambda + \Lambda^\sim Q + Q^\sim \Pi Q = 0 \quad (39)$$

for almost all $s \in \mathbb{C}$, where

$$\Phi := U^\sim J_\mu U - V^\sim J_\mu V, \quad (40)$$

$$\Lambda := M^\sim J_\mu U - N^\sim J_\mu V, \quad (41)$$

$$\Pi := M^\sim J_\mu M - N^\sim J_\mu N. \quad (42)$$

Furthermore, under the condition (37), the feedthrough matrix $K(\infty)$ is well-defined and inherits the (J_μ, J_μ) -unitary condition from K .

PROOF. The (J_μ, J_μ) -unitary condition for the controller

$$K^\sim(s)J_\mu K(s) = J_\mu, \quad (43)$$

which must be satisfied for all $s \in \mathbb{C}$, can be represented in terms of the right factorization from (36) as

$$((U+MQ)(V+NQ)^{-1})^\sim J_\mu (U+MQ)(V+NQ)^{-1} = J_\mu. \quad (44)$$

Using the properties of system conjugation, (44) is equivalent to

$$(U+MQ)^\sim J_\mu (U+MQ) = (V+NQ)^\sim J_\mu (V+NQ).$$

After regrouping terms, this equality takes the form

$$U^\sim J_\mu U - V^\sim J_\mu V + Q^\sim (M^\sim J_\mu U - N^\sim J_\mu V) + (U^\sim J_\mu M - V^\sim J_\mu N)Q + Q^\sim (M^\sim J_\mu M - N^\sim J_\mu N)Q = 0.$$

This leads to (39), with Φ , Λ , Π given by (40)–(42). The fact that the condition (37) makes the feedthrough matrix $K(\infty)$ well-defined follows directly from (36). The (J_μ, J_μ) -unitarity of $K(\infty)$ is established by taking the limit in (43) as $s \rightarrow \infty$. ■

Since the (J_μ, J_μ) -unitary condition (43) and its equivalent dual form $KJ_\mu K^\sim = J_\mu$ (cf. (23) and (24)) impose the same constraints on the square transfer matrix K , a dual condition to the one described in Lemma 6 holds for the left factorization of the controller in (36). This leads to a dual constraint on Q , which corresponds to (39), with Φ , Λ , Π being replaced with their counterparts expressed in terms of \hat{N} , \hat{M} , \hat{U} , \hat{V} . In Appendix E, we also show how expression (44) imposes constraints on the state-space realization of the Youla parameter.

Theorem 7 Suppose the block P_{22} of the plant transfer matrix P in (25) has the coprime factorizations over \mathcal{RH}_∞ described by (28). Also, let the transfer matrices $U, V, \hat{U}, \hat{V} \in \mathcal{RH}_\infty$ in (29) satisfy the general Bézout identity (32). Then the set of all stabilizing (J_μ, J_μ) -unitary controllers K with a well-defined feedthrough matrix $K(\infty)$ is parameterized by (36), where the parameter Q belongs to the set

$$\mathcal{Q} := \{Q \in \mathcal{RH}_\infty \text{ satisfying (37) and (39)}\}. \quad (45)$$

PROOF. This theorem is proved by combining Lemmas 5 and 6. Indeed, since the underlying coprime factorizations are assumed to satisfy the general Bézout identity (32), then (39) can be applied to the common parameter Q in (36) in order to describe all stabilizing (J_μ, J_μ) -unitary controllers K . Their feedthrough matrices $K(\infty)$ are well-defined provided the additional condition (37) is also satisfied. The resulting class of admissible Q is given by (45). ■

Theorem 7 provides a frequency domain parameterization of all stabilizing (J_μ, J_μ) -unitary controllers with a well-defined feedthrough matrix and leads to the following theorem.

Theorem 8 Under the assumptions of Theorem 7, the MFDs (36) describe the set of all stabilizing PR quantum controllers K , where the parameter Q belongs to the following class $\hat{\mathcal{Q}}$ defined in terms of (45):

$$\hat{\mathcal{Q}} := \{Q \in \mathcal{Q} : K(\infty) \in \mathcal{O}(\mu)\}. \quad (46)$$

PROOF. The assertion of the theorem is established by combining Theorem 7 with the frequency domain criterion of PR provided by Lemma 2.

The result of Theorem 8 provides the parameterization of all stabilizing linear coherent quantum controllers in the frequency domain.

In view of the results of Lemma 6, the constraint (39) on the Youla-Kučera parameter Q inherits its quadratic nature from (43). Equation (39) becomes affine (over the field of reals) with respect to Q in a particular case when $\Pi = 0$. In view of (28), the

transfer function Π in (42) is representable as $\Pi = \tilde{M}^*(J_\mu - P_{22}^* J_\mu P_{22})M$, and hence, it vanishes if the block P_{22} of the plant is (J_μ, J_μ) -unitary. The following example shows that there exists a PR plant P such that its block P_{22} is (J_μ, J_μ) -unitary.

Example 9 Suppose P , defined in (25), represents a one-mode LQSS with the associated (D, M, R) parameterization (20) and dimension $m = 4$. Also, assume that the corresponding matrix $D = \text{diag}(D_{11}, D_{22})$ and $B_1 J_2 B_1^T = 0$ ($\det B_1 = 0$). It can be shown by inspection that the block P_{22} , given in (27), also represents a LQSS.

However, in view of the results of the following lemma, Corollary 11 shows that the corresponding feedback connection in Fig. 3 cannot be stabilized by a linear coherent quantum controller.

Lemma 10 Suppose the block P_{22} of the plant transfer matrix P in (25) is a (J_μ, J_μ) -unitary system. Then there exists no (J_μ, J_μ) -unitary controller K that can stabilize the system in Fig. 5.

PROOF. The system matrix \mathbf{A} for the closed-loop system in Fig. 5 can be calculated as

$$\mathbf{A} = \begin{bmatrix} A + B_2 \Delta d C_2 & B_2 \Delta c \\ b D_{22} \Delta d C_2 + b C_2 & a + b D_{22} \Delta c \end{bmatrix}, \quad (47)$$

where the minimal realizations of the block P_{22} and the controller K are $P_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array} \right]$ and $K = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array} \right]$. Also, $\Delta := (I - d D_{22})^{-1}$. Since P_{22} and K are (J_μ, J_μ) -unitary, there exist unique non-singular matrices $\Theta_1 \in \mathbb{A}_{n_1}$ and $\Theta_2 \in \mathbb{A}_{n_2}$, where n_1 and n_2 are the McMillan degree of P_{22} and K [32] such that:

$$C_2 = D_{22} J_\mu B_2^T \Theta_1, \quad 0 = A^T \Theta_1 + \Theta_1 A + C_2^T J_\mu C_2, \quad (48)$$

$$c = d J_\mu b^T \Theta_2, \quad 0 = a^T \Theta_2 + \Theta_2 a + c^T J_\mu c. \quad (49)$$

Then, in view of (48) and (49),

$$\begin{aligned} \mathbf{A}^T \Theta + \Theta \mathbf{A} &= \begin{bmatrix} A + B_2 \Delta d C_2 & B_2 \Delta c \\ b D_{22} \Delta d C_2 + b C_2 & a + b D_{22} \Delta c \end{bmatrix}^T \Theta + \Theta \begin{bmatrix} A + B_2 \Delta d C_2 & B_2 \Delta c \\ b D_{22} \Delta d C_2 + b C_2 & a + b D_{22} \Delta c \end{bmatrix} \\ &= \begin{bmatrix} \Theta_1 B_2 \Xi B_2^T \Theta_1 & \Theta_1 B_2 \Xi D_{22}^T b^T \Theta_2 \\ \Theta_2 b D_{22} \Xi B_2^T \Theta_1 & \Theta_2 b D_{22} \Xi D_{22}^T b^T \Theta_2 \end{bmatrix}, \end{aligned} \quad (50)$$

where $\Theta := \text{diag}(\Theta_1, \Theta_2)$, $\Xi := J_\mu + \Delta d D_{22} J_\mu + J_\mu D_{22}^T d^T \Delta^T$, and use is made of

$$D_{22}^T J_\mu D_{22} = D_{22} J_\mu D_{22}^T = J_\mu, \quad d^T J_\mu d = d J_\mu d^T = J_\mu.$$

Also, multiplying by Ξ on the left and on the right by Δ^{-1} and Δ^{-T} , respectively, we have

$$\begin{aligned} \Delta^{-1} \Xi \Delta^{-T} &= (I - d D_{22}) J_\mu (I - d D_{22})^T + d D_{22} J_\mu (I - d D_{22})^T + (I - d D_{22}) J_\mu D_{22}^T d^T \\ &= J_\mu - d D_{22} J_\mu - J_\mu D_{22}^T d^T + J_\mu + d D_{22} J_\mu - J_\mu + J_\mu D_{22}^T d^T - J_\mu \\ &= 0. \end{aligned}$$

This implies that Ξ and, consequently, the right-hand side of (50) is zero. Then there exists a matrix $R \in \mathbb{S}_{n_1+n_2}$ for which $\mathbf{A} = \Theta R$. Since \mathbf{A} is similar to a Hamiltonian matrix, its spectrum is symmetric about the imaginary axis; therefore, the closed-loop system in Fig. 5 cannot be asymptotically stable. \blacksquare

Corollary 11 Suppose the block P_{22} of the plant transfer matrix P in (25) is a (J_μ, J_μ) -unitary system. Then there exists no linear coherent quantum controller K that can stabilize the system in Fig. 3.

PROOF. In view of the results of Lemma 10, the assertion of the corollary is established by combining Theorem 8 with the results of Lemma 3. ■

Corollary 11 can be considered as a no-go result for linear quantum system stabilization.

8 Coherent Quantum Weighted \mathcal{H}_2 and \mathcal{H}_∞ Control Problems in the Frequency Domain

The following lemma, which is given here for completeness, employs the factorization approach in order to obtain a more convenient representation of the closed-loop transfer function when the controller is represented in terms of the Youla-Kučera parameterization.

Lemma 12 *Under the assumptions of Theorem 7, for any stabilizing controller K parameterized by (36), the corresponding closed-loop transfer matrix G in (26) is representable as*

$$G = T_0 + T_1 Q T_2, \quad (51)$$

where

$$T_0 := P_{11} + P_{12} U \hat{M} P_{21}, \quad T_1 := P_{12} M, \quad T_2 := \hat{M} P_{21}. \quad (52)$$

PROOF. By substituting P_{22} from (28) and K from (36) into (26), it follows that

$$\begin{aligned} G &= P_{11} + P_{12}(U + MQ)(V + NQ)^{-1}(I - NM^{-1}(U + MQ)(V + NQ)^{-1})^{-1}P_{21} \\ &= P_{11} + P_{12}(U + MQ)(V + NQ - NM^{-1}(U + MQ))^{-1}P_{21} \\ &= P_{11} + P_{12}(U + MQ)(V - NM^{-1}U)^{-1}P_{21} \\ &= P_{11} + P_{12}(U + MQ)\hat{M}P_{21}, \end{aligned}$$

which leads to (51) and (52). Here, use is made of the relations $V - NM^{-1}U = V - \hat{M}^{-1}\hat{N}U = V - \hat{M}^{-1}(\hat{M}V - I) = \hat{M}^{-1}$ which are obtained from (28) and (29). ■

Lemma 12 allows the following coherent quantum weighted \mathcal{H}_2 and \mathcal{H}_∞ control problems to be formulated in the frequency domain.

8.1 Coherent Quantum Weighted \mathcal{H}_2 Control Problem

Using the representation (51), we formulate a coherent quantum weighted \mathcal{H}_2 control problem as the constrained minimization problem

$$E := \|W_{\text{out}} G W_{\text{in}}\|_2^2 = \|T_0 + T_1 Q T_2\|_2^2 \longrightarrow \min \quad (53)$$

with respect to $Q \in \hat{\mathcal{Q}}$, where the set $\hat{\mathcal{Q}}$ is given by (46). Here,

$$T_0 := W_{\text{out}} T_0 W_{\text{in}}, \quad T_1 := W_{\text{out}} T_1, \quad T_2 := T_2 W_{\text{in}}, \quad (54)$$

where T_0, T_1, T_2 are defined by (52). Also, $W_{\text{in}}, W_{\text{out}} \in \mathcal{RH}_\infty$ are given weighting transfer functions for the closed-loop system G which ensure that $T_0 + T_1 Q T_2 \in \mathcal{RH}_2$. The \mathcal{H}_2 -norm $\|\cdot\|_2$ is associated with the inner product $\langle \Gamma_1, \Gamma_2 \rangle := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle \Gamma_1(i\omega), \Gamma_2(i\omega) \rangle_F d\omega$, where $\langle \cdot, \cdot \rangle_F$ refers to the Frobenius inner product of complex matrices. By using the standard properties of inner products in complex Hilbert spaces [30], the cost functional E in (53) can be represented as

$$E = \|T_0\|_2^2 + 2\text{Re}\langle \hat{T}_0, Q \rangle + \langle Q, \hat{T}_1 Q \hat{T}_2 \rangle, \quad (55)$$

where

$$\hat{T}_0 := T_1^\sim T_0 T_2^\sim, \quad \hat{T}_1 := T_1^\sim T_1, \quad \hat{T}_2 := T_2 T_2^\sim. \quad (56)$$

In comparison to the original coherent quantum LQG (CQLQG) control problem [25], the coherent quantum weighted \mathcal{H}_2 control problem (53) allows for a frequency domain weighting of the cost.

8.2 Coherent Quantum Weighted \mathcal{H}_∞ Control Problem

Similarly to (53), a coherent quantum weighted \mathcal{H}_∞ control problem is formulated as the constrained minimization problem

$$\|G\|_\infty = \|T_0 + T_1 Q T_2\|_\infty \longrightarrow \min \quad (57)$$

with respect to $Q \in \hat{\mathcal{D}}$, where the set $\hat{\mathcal{D}}$ is defined by (46). Here, T_0 , T_1 and T_2 are given by (54), where, this time, the weighting transfer functions $W_{\text{in}}, W_{\text{out}} \in \mathcal{RH}_\infty$ are not necessarily strictly proper. Recall that the norm in the Hardy space \mathcal{H}_∞ is defined by $\|\Gamma\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(\Gamma(i\omega))$, where $\sigma_{\max}(\cdot)$ denotes the largest singular value of a matrix. Note that both problems (53) and (57) are organised as constrained versions of the model matching problem [9]. Since the \mathcal{H}_2 control problem is based on a Hilbert space norm, its solution can be approached by using a variational method in the frequency domain, which employs differentiation of the cost E with respect to the Youla-Kučera parameter Q and is qualitatively different from the state-space techniques of [35].

9 Projected Gradient Descent Scheme for the Coherent Quantum Weighted \mathcal{H}_2 Control Problem

Suppose the set $\hat{\mathcal{D}}$ in (46) is nonempty, and hence, there exist stabilizing PR quantum controllers for a given quantum plant. Tractable conditions for the existence of such controllers remain an open problem which is not considered here. By using the representation (55) and regarding the transfer function $Q \in \mathcal{RH}_\infty$ as an independent optimization variable, it follows that the first variation of the cost functional E in (53) with respect to Q can be computed as

$$\delta E = \text{Re}\langle \partial E, \delta Q \rangle, \quad \partial E := 2(\hat{T}_0 + \hat{T}_1 Q \hat{T}_2), \quad (58)$$

where use is also made of (56). In order to yield a PR quantum controller, Q must satisfy the constraint (39) whose variation leads to

$$\delta Q^\sim (\Lambda + \Pi Q) + (\Lambda^\sim + Q^\sim \Pi) \delta Q = 0. \quad (59)$$

In view of the uniqueness theorem for analytic functions [22], the resulting constrained optimization problem can be reduced to that for purely imaginary $s = i\omega$ (with an assumption of analyticity in a strip which includes the imaginary axis for the transfer functions involved), with $\omega \in \mathbb{R}$. The transfer matrices δQ , satisfying (59) at frequencies ω from a given set $\Omega \subset \mathbb{R}$, form a real subspace of transfer functions

$$\mathcal{S} := \{X \in \mathcal{RH}_2 : (X^*(\Lambda + \Pi Q) + (\Lambda^* + Q^* \Pi)X)|_{i\Omega} = 0\}. \quad (60)$$

For practical purposes, the set Ω is used to “discretize” the common frequency range of the given weighting transfer functions $W_{\text{in}}, W_{\text{out}}$ in the coherent quantum weighted \mathcal{H}_2 control problem (53). A conceptual solution of this problem can be implemented in the form of the following projected gradient descent scheme for finding a critical point of the cost functional E with respect to Q subject to (39) at a finite set of frequencies Ω :

- (1) initialize $Q \in \mathcal{RH}_\infty$ so as to satisfy (39), which yields a stabilizing PR quantum controller; choose the input and output weights $W_{\text{in}}(i\omega)$ and $W_{\text{out}}(i\omega)$, and assign a discrete frequency array ω in the set $[-\omega_\ell, \omega_u] \subseteq \mathbb{R}$. Set the step size $\alpha > 0$.
- (2) calculate $\partial E(i\omega)$ according to (58) for each frequency $\omega \in \Omega$;
- (3) compute $\delta Q(i\omega) = -\alpha \text{Proj}_{\mathcal{S}}(\partial E(i\omega))$ by using a projection onto the set \mathcal{S} and a parameter $\alpha > 0$;
- (4) update Q to $Q + \delta Q$, and go to the second step.

The gradient projection $\text{Proj}_{\mathcal{S}}(\partial E)$ onto the set \mathcal{S} in (60) is computed in the third step of the algorithm by solving a convex optimization problem; see Appendix F for more information. This algorithm also involves interpolation constraints on transfer functions; see [2, 15] for more details. The discrete frequency set Ω and the step-size parameter α can be chosen adaptively at each iteration of the algorithm. The outcome of the algorithm is considered to be acceptable if Q belongs to the set $\hat{\mathcal{D}}$ defined by (46) of Theorem 8.

10 Conclusion

In view of the PR constraints, the set of all stabilizing linear coherent quantum controllers for a given linear quantum plant has been parameterized using a Youla-Kučera factorization approach. This approach has provided a formulation of the coherent quantum weighted \mathcal{H}_2 and \mathcal{H}_∞ control problems for linear quantum systems in the frequency domain. These problems resemble constrained versions of the classical model matching problem. The proposed framework can also be used to develop tractable conditions for the existence of stabilizing quantum controllers for a given quantum plant, which remains an open problem. This problem is also important for the generation of stabilizing PR quantum controllers as initial approximations for iterative algorithms of CQLQG control design. By developing a numerical algorithm for solving the coherent quantum weighted \mathcal{H}_2 or \mathcal{H}_∞ control problems, the results of this paper can be used to solve the control problems for real physical systems. A numerical algorithm for solving the weighted \mathcal{H}_2 problem can be designed based on the conceptual scheme proposed in this paper. This is a subject for further research and will be addressed in future.

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A Parameterization of Linear Coherent Quantum Feedback Systems in Position-Momentum Form

The following lemma shows the feedback system in Fig. 3 represents a LQSS.

Lemma 13 *Suppose the LQSSs P and K (with vectors of dynamic variables X_1 and X_2 satisfying the CCRs (18)) are interconnected as shown in Fig. 3. Then the resulting closed-loop system is also an LQSS.*

In what follows, we provide a proof for Lemma 13. In particular, we consider the plant and the controller parameterized by the triples (D_1, M_1, R_1) and (D_2, M_2, R_2) . Then we show that the closed-loop feedback system in Fig. 3 represents a LQSS which can be parameterized by the triple (D, M, R) :

$$D = D_{11} + D_{12}\Delta D_2 D_{21}, \quad (A1)$$

$$M = -\frac{1}{2}B^T\Theta^{-1}, \quad (A2)$$

$$R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} + \widehat{R}, \quad (A3)$$

where the matrices $B_1 := 2\Theta_1 M_1^T$ and D_1 are partitioned as $B_1 =: \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$, $D_1 =: \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$,

$$\Delta := (I - D_2 D_{22})^{-1}, \quad B := \begin{bmatrix} B_{11} + B_{12}\Delta D_2 D_{21} \\ B_2(D_{21} + D_{22}\Delta D_2 D_{21}) \end{bmatrix}, \quad B_2 := 2\Theta_2 M_2^T. \quad (A4)$$

Also, the block entries of the symmetric matrix \widehat{R} are formulated as

$$\begin{aligned}\widehat{R}_{11} &= \frac{1}{4}\Theta_1^{-1}\left(B_{12}\Delta D_2 D_{21} J B_{11}^T - B_{11} J D_{21}^T D_2^T \Delta^T B_{12}^T \right. \\ &\quad \left. + B_{12}\Delta D_2 D_{22} J B_{12}^T - B_{12} J D_{22}^T D_2^T \Delta^T B_{12}^T\right)\Theta_1^{-T}, \\ \widehat{R}_{12} &= \frac{1}{4}\Theta_2^{-1}B_2\left(D_{22}\Delta D_2 D_{21} J B_{11}^T + D_{21} J B_{11}^T + D_{22}\Delta D_2 D_{22} J B_{12}^T \right. \\ &\quad \left. + D_{22} J B_{12}^T - J D_2^T \Delta^T B_{12}^T\right)\Theta_1^{-T}, \\ \widehat{R}_{22} &= \frac{1}{4}\Theta_2^{-1}B_2\left(D_{22}\Delta D_2 J - J D_2^T \Delta^T D_{22}^T\right)B_2^T \Theta_2^{-T}.\end{aligned}$$

PROOF. The matrices A , B , C and D for the closed-loop system can be calculated as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 + B_{12}\Delta D_2 C_{21} & B_{12}\Delta C_2 & \vdots & B_{11} + B_{12}\Delta D_2 D_{21} \\ -\frac{B_2 D_{22}\Delta D_2 C_{21} + B_2 C_{21}}{C_{11} + D_{12}\Delta D_2 C_{12}} & -\frac{A_2 + B_2 D_{22}\Delta C_2}{D_{12}\Delta C_2} & \vdots & -\frac{B_2 D_{21} + B_2 D_{22}\Delta D_2 D_{21}}{D_{11} + D_{12}\Delta D_2 D_{21}} \end{bmatrix}, \quad (\text{A5})$$

where the state-space realizations for the plant and the controller

$$P = \begin{bmatrix} A_1 & B_{11} & B_{12} \\ C_{11} & D_{11} & D_{12} \\ C_{21} & D_{21} & D_{22} \end{bmatrix}, \quad K = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix},$$

are the corresponding parameters for the plant and the controller, which depend on the associated Hamiltonian and coupling parameterizations of these systems, and Δ is defined in (A4).

We will show that the realization (A5) is associated with a LQSS. In the first step, we show that the matrix D is an orthogonal symplectic matrix, that is,

$$D(I + iJ)D^T = I + iJ. \quad (\text{A6})$$

In (A6) and in what follows the subscripts in the matrices I_k and J_k of order k are omitted for brevity. The matrices D_1 and D_2 are the feedthroughs of the LQSSs P and K ; therefore, it can be shown that

$$D_{11}(I + iJ)D_{11}^T + D_{12}(I + iJ)D_{12}^T = I + iJ, \quad (\text{A7})$$

$$D_{11}(I + iJ)D_{21}^T + D_{12}(I + iJ)D_{22}^T = 0, \quad (\text{A8})$$

$$D_{21}(I + iJ)D_{21}^T + D_{22}(I + iJ)D_{22}^T = I + iJ, \quad (\text{A9})$$

$$D_2(I + iJ)D_2^T = I + iJ. \quad (\text{A10})$$

In view of (A5), from the left hand side of (A6), it follows that

$$\begin{aligned}D(I + iJ)D^T &= (D_{11} + D_{12}\Delta D_2 D_{21})(I + iJ)(D_{11} + D_{12}\Delta D_2 D_{21})^T \\ &= D_{11}(I + iJ)D_{11}^T \\ &\quad + D_{11}(I + iJ)D_{21}^T D_2^T \Delta^T D_{12}^T + D_{12}\Delta D_2 D_{21}(I + iJ)D_{11}^T \\ &\quad + D_{12}\Delta D_2 D_{21}(I + iJ)D_{21}^T D_2^T \Delta^T D_{12}^T \\ &= (I + iJ) - D_{12}(I + iJ)D_{12}^T \\ &\quad - D_{12}(I + iJ)D_{22}^T D_2^T \Delta^T D_{12}^T - D_{12}\Delta D_2 D_{22}(I + iJ)D_{12}^T \\ &\quad + D_{12}\Delta D_2 \left((I + iJ) - D_{22}(I + iJ)D_{22}^T\right)D_2^T \Delta^T D_{12}^T \\ &= I + iJ,\end{aligned}$$

where use is also made of the following equation

$$\begin{aligned} 0 = & -(I + iJ) - (I + iJ)D_{22}^T D_2^T \Delta^T - \Delta D_2 D_{22}(I + iJ) \\ & + \Delta D_2 \left((I + iJ) - D_{22}(I + iJ)D_{22}^T \right) D_2^T \Delta^T, \end{aligned} \quad (\text{A11})$$

which, in view of (A10), can be proved by multiplying its right-hand side from left and right by $\Delta^{-1} = I - D_2 D_{22}$ and Δ^{-T} .

In the second step, we will prove that the non-demolition nature of the output fields of the closed-loop system is preserved. That is,

$$C\Theta + DJB^T = 0, \quad (\text{A12})$$

where $\Theta := \text{diag}(\Theta_1, \Theta_2)$. The non-demolition conditions for outputs of the plant and the controller imply

$$\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = - \begin{bmatrix} (D_{11}JB_{11}^T + D_{12}JB_{12}^T)\Theta_1^{-1} \\ (D_{21}JB_1^T + D_{22}JB_2^T)\Theta_1^{-1} \end{bmatrix}, \quad (\text{A13})$$

$$0 = C_2\Theta_2 + D_2JB_2^T. \quad (\text{A14})$$

Then from the left hand side of the condition (A12)

$$\begin{aligned} C\Theta + DJB^T &= \begin{bmatrix} C_{11}\Theta_1 + D_{12}\Delta D_2 C_{21}\Theta_1 \\ D_{12}\Delta C_2\Theta_2 \end{bmatrix} + (D_{11} + D_{12}\Delta D_2 D_{21})J \times \\ &\quad \begin{bmatrix} B_{11}^T + D_{21}^T D_2^T \Delta^T B_{12}^T \\ (D_{21}^T + D_{21}^T D_2^T \Delta^T D_{22}^T)B_2^T \end{bmatrix} \\ &= \begin{bmatrix} -(D_{11}JB_{11}^T + D_{12}JB_{12}^T) - D_{12}\Delta D_2 (D_{21}JB_{11}^T + D_{22}JB_{12}^T) \\ -D_{12}\Delta D_2 JB_{12}^T \end{bmatrix} \\ &\quad + (D_{11} + D_{12}\Delta D_2 D_{21})J \begin{bmatrix} B_{11}^T + D_{21}^T D_2^T \Delta^T B_{12}^T \\ (D_{21}^T + D_{21}^T D_2^T \Delta^T D_{22}^T)B_2^T \end{bmatrix} \\ &= D_{12}(-J - \Delta d D_{22}J - JD_{22}^T \Delta^T + \Delta D_2(J - D_{22}JD_{22}^T)D_2^T \Delta^T) \begin{bmatrix} B_{12}^T \\ D_{22}^T B_2^T \end{bmatrix} \\ &= 0, \end{aligned}$$

where use is made of (A8), (A9), the real part of (A11), (A13) and (A14).

In the final step, in view of (13), we will compute the Hamiltonian parameter R of the closed-loop system from

$$\hat{R} = \frac{1}{2} \left(\Theta^{-1}A - \frac{1}{2}\Theta^{-1}BJB^T\Theta^{-T} \right). \quad (\text{A15})$$

The realization in (A5) implies

$$\begin{aligned} E &:= \frac{1}{2} \Theta^{-1} BJB^T \Theta^{-T} \\ &= \frac{1}{2} \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}^{-1} \begin{bmatrix} B_{11} + B_{12}\Delta D_2 D_{21} \\ B_2 D_{21} + B_2 D_{22}\Delta D_2 D_{21} \end{bmatrix} J \begin{bmatrix} B_{11} + B_{12}\Delta D_2 D_{21} \\ B_2 D_{21} + B_2 D_{22}\Delta D_2 D_{21} \end{bmatrix}^T \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}^{-T}, \end{aligned}$$

where the block entries of the matrix E can be calculated as

$$E_{11} = \frac{1}{2}\Theta_1^{-1}(B_{11}JB_{11}^T + B_{11}JD_{21}^TD_2^T\Delta^TB_{12}^T + B_{12}\Delta D_2D_{21}JB_{11}^T + B_{12}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TB_{12}^T)\Theta_1^{-T}, \quad (A16)$$

$$E_{12} = \frac{1}{2}\Theta_1^{-1}(B_{11}JD_{21}^T + B_{11}JD_{21}^TD_2^T\Delta^TD_{22}^T + B_{12}\Delta D_2D_{21}JD_{21}^T + B_{12}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TD_{22}^T)B_2^T\Theta_2^{-T}, \quad (A17)$$

$$E_{22} = \frac{1}{2}\Theta_2^{-1}B_2(D_{21}JD_{21}^T + D_{21}JD_{21}^TD_2^T\Delta^TD_{22}^T + D_{22}\Delta D_2D_{21}JD_{21}^T + D_{22}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TD_{22}^T)B_2^T\Theta_2^{-T} \quad (A18)$$

and $E_{12} = -E_{21}^T$. It follows from (A15) that the closed-loop Hamiltonian parameter is given by

$$R = \frac{1}{2} \begin{bmatrix} \Theta_1^{-1}A_1 + \Theta_1^{-1}B_{12}\Delta D_2C_{21} & \Theta_1^{-1}B_{12}\Delta C_2 \\ \Theta_2^{-1}B_2D_{22}\Delta D_2C_{21} + B_2C_{21} & \Theta_2^{-1}A_2 + \Theta_2^{-1}B_2D_{22}\Delta C_2 \end{bmatrix} - \frac{1}{2}E. \quad (A19)$$

Then the one-one block entry of R can be calculated as

$$\begin{aligned} R_{11} &= R_1 - \frac{1}{4}\Theta_1^{-1} \left(-B_{11}JB_{11}^T - B_{12}JB_{12}^T \right. \\ &\quad - 2B_{12}\Delta D_2(D_{21}JB_{11}^T + D_{22}JB_{12}^T) \\ &\quad + B_{11}JB_{11}^T + B_{11}JD_{21}^TD_2^T\Delta^TB_{12}^T + B_{12}\Delta D_2D_{21}JB_{11}^T \\ &\quad \left. + B_{12}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TB_{12}^T \right) \Theta_1^{-T} \\ &= R_1 - \frac{1}{4}\Theta_1^{-1} \left(-B_{12}JB_{12}^T - B_{12}\Delta D_2D_{21}JB_{11}^T - 2B_{12}\Delta D_2D_{22}JB_{12}^T \right. \\ &\quad \left. + B_{12}JD_{21}^TD_2^T\Delta^TB_{12}^T + B_{12}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TB_{12}^T \right) \Theta_1^{-T} \\ &= R_1 - \frac{1}{4}\Theta_1^{-1} \left(-B_{12}\Delta D_2D_{21}JB_{11}^T + B_{11}JD_{21}^TD_2^T\Delta^TB_{12}^T \right. \\ &\quad - B_{12}\Delta D_2D_{22}JB_{12}^T + B_{12}JD_{22}^TD_2^T\Delta^TB_{12}^T \\ &\quad + B_{12}(-J - \Delta D_2D_{22}J - JD_{22}^TD_2^T\Delta^T \\ &\quad \left. + \Delta D_2D_{21}JD_{21}^TD_2^T\Delta^T)B_{12}^T \right) \Theta_1^{-T} \\ &= R_1 + \frac{1}{4}\Theta_1^{-1} \left(B_{12}\Delta D_2D_{21}JB_{11}^T - B_{11}JD_{21}^TD_2^T\Delta^TB_{12}^T \right. \\ &\quad \left. + B_{12}\Delta D_2D_{22}JB_{12}^T - B_{12}JD_{22}^TD_2^T\Delta^TB_{12}^T \right) \Theta_1^{-T}, \end{aligned}$$

where use is made of the dependence of A_k on the parameters (D_k, M_k, R_k) for $k = 1, 2$, (A11), (A13) and the fact that Θ is a skew-symmetric matrix. Similarly, it can be shown that the two-two block of R is

$$\begin{aligned} R_{22} &= R_2 - \frac{1}{4}\Theta_2^{-1}B_2(-J - 2D_{22}\Delta D_2J + D_{21}JD_{21}^T + D_{21}JD_{21}^TD_2^T\Delta^TD_{22}^T \\ &\quad + D_{22}\Delta D_2D_{21}JD_{21}^T + D_{22}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TD_{22}^T)B_2^T\Theta_2^{-T} \\ &= R_2 + \frac{1}{4}\Theta_2^{-1}B_2(D_{22}\Delta D_2J - JD_{22}^T\Delta^TD_{22}^T)B_2^T\Theta_2^{-T}. \end{aligned}$$

The off-diagonal terms of the closed-loop Hamiltonian parameter are computed as

$$\begin{aligned} R_{12} &= \frac{1}{4}\Theta_1^{-1}(2B_{12}\Delta D_2J - B_{11}JD_{21}^T - B_{11}JD_{21}^TD_2^T\Delta^TD_{22}^T - B_{12}\Delta D_2D_{21}JD_{21}^T - B_{12}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TD_{22}^T)B_2^T\Theta_2^{-T}, \\ &= \frac{1}{4}\Theta_1^{-1}(B_{12}\Delta D_2J - B_{11}JD_{21}^T - B_{11}JD_{21}^TD_2^T\Delta^TD_{22}^T + B_{12}\Delta D_2(J - D_{21}JD_{21}^T - D_{21}JD_{21}^TD_2^T\Delta^TD_{22}^T))B_2^T\Theta_2^{-T}, \\ &= \frac{1}{4}\Theta_1^{-1}(B_{12}\Delta D_2J - B_{11}JD_{21}^T - B_{11}JD_{21}^TD_2^T\Delta^TD_{22}^T + B_{12}(\Delta D_2D_{22}J - \Delta D_2(J - D_{22}JD_{22}^T)D_2^T\Delta^T)D_{22}^T)B_2^T\Theta_2^{-T}, \\ &= \frac{1}{4}\Theta_1^{-1}(B_{12}\Delta D_2J - B_{11}JD_{21}^T - B_{11}JD_{21}^TD_2^T\Delta^TD_{22}^T - B_{12}JD_{22}^T - B_{12}JD_{22}^TD_2^T\Delta^TD_{22}^T)B_2^T\Theta_2^{-T}, \end{aligned}$$

$$\begin{aligned}
R_{21} &= \frac{1}{4}\Theta_2^{-1}B_2(D_{22}\Delta D_2D_{21}JB_{11}^T + 2D_{22}\Delta D_2D_{22}JB_{12}^T \\
&\quad + D_{21}JB_{11}^T + 2D_{22}JB_{12}^T - D_{21}JD_{21}^TD_2^T\Delta^TB_{12}^T \\
&\quad - D_{22}\Delta D_2D_{21}JD_{21}^TD_2^T\Delta^TB_{12}^T)\Theta_1^{-T} \\
&= \frac{1}{4}\Theta_2^{-1}B_2(D_{22}\Delta D_2D_{21}JB_{11}^T + D_{21}JB_{11}^T \\
&\quad + 2D_{22}\Delta D_2D_{22}JB_{12}^T + 2D_{22}JB_{12}^T - D_{21}JD_{21}^TD_2^T\Delta^TB_{12}^T \\
&\quad - D_{22}(J + JD_{22}^TD_2^T\Delta^T + \Delta D_2D_{22}J)B_{12}^T)\Theta_1^{-T} \\
&= \frac{1}{4}\Theta_2^{-1}B_2(D_{22}\Delta D_2D_{21}JB_{11}^T + D_{21}JB_{11}^T \\
&\quad + D_{22}\Delta D_2D_{22}JB_{12}^T + D_{22}JB_{12}^T - JD_2^T\Delta^TB_{12}^T)\Theta_1^{-T},
\end{aligned}$$

where use is made of (A11). It can be shown, by inspection, that $R_{12} = R_{21}^T$ and, as a result, the matrix R is symmetric. Then the corresponding (D, M, R) parametrization for the closed-loop system can be formulated as in (A1)–(A3). ■

B Cholesky-like Factorizations for Skew-Symmetric Matrices

The existence of Cholesky-like factorizations is addressed in the following lemma.

Lemma 14 *Let $\Theta \in \mathbb{A}_n$ be a non-singular matrix. Then there exists a non-singular matrix $\Sigma \in \mathbb{R}^{n \times n}$ such that $\Theta = \Sigma J_n \Sigma^T$.*

PROOF. As a consequence of spectral decomposition, in the Murnaghan canonical form (see [4] and the references therein), there exists a factorization $\Theta = O\Delta O^T$, where the matrix $O \in \mathbb{R}^{n \times n}$ is orthogonal and the matrix $\Delta \in \mathbb{R}^{n \times n}$ is block diagonal. Each block on the main diagonal of the matrix Δ has the form $\begin{bmatrix} 0 & \delta_i \\ -\delta_i & 0 \end{bmatrix}$ with $\delta_i > 0$, where $\pm i\delta_i$ is a pair of complex conjugate eigenvalues of Θ . Then, there exists a decomposition $\Theta = \Sigma J_n \Sigma^T$, where the matrix $\Sigma = O \text{diag}\{\sqrt{\delta_1}, \sqrt{\delta_1}, \dots, \sqrt{\delta_n}, \sqrt{\delta_n}\}$ is non-singular. Also, for any such Σ , the matrix $\Sigma \hat{\Sigma}^T$ leads to the decomposition of Θ , where $\hat{\Sigma} \in \text{Sp}(n, \mathbb{R})$. ■

In view of Lemma 14, any two non-singular matrices $\Theta_1, \Theta_2 \in \mathbb{A}_n$ are related to each other by a non-singular matrix $\hat{\Sigma}$ as $\Theta_1 = \hat{\Sigma}\Theta_2\hat{\Sigma}^T$, where $\hat{\Sigma} = \Sigma_1\Sigma_2^{-1}$ and $\Theta_k = \Sigma_k J_n \Sigma_k^T$ for $k = 1, 2$.

C Lemmas on Linear Fractional Transformation

The following lemmas provide relationships between the MFDs and LFT representation of stabilizing controllers in Sections 6 and 7.

Lemma 15 [40] *Suppose V is invertible. Then the following MFDs are represented as LFTs:*

$$\begin{aligned}
(U + MQ)(V + NQ)^{-1} &= \text{LFT}(O_y, Q), \\
(V + QN)^{-1}(U + QM) &= \text{LFT}(O_z, Q),
\end{aligned}$$

where O_y and O_z are auxiliary systems given by

$$\begin{aligned}
O_y &:= \begin{bmatrix} UV^{-1} & M - UV^{-1}N \\ V^{-1} & -V^{-1}N \end{bmatrix}, \\
O_z &:= \begin{bmatrix} V^{-1}U & V^{-1} \\ M - NV^{-1}U & -NV^{-1} \end{bmatrix}.
\end{aligned} \tag{C1}$$

The converse of Lemma 15 also holds under certain conditions on the system O_y which are addressed below.

Lemma 16 [40] Suppose the system O_y is partitioned as $O_y := \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}$. Then the following LFTs are represented as MFDs:

(a) if O_{21} is invertible, then

$$\text{LFT}(O_y, Q) = (U + MQ)(V + NQ)^{-1},$$

with

$$\begin{aligned} U &= O_{11}O_{21}^{-1}, & M &= O_{12} - O_{11}O_{21}^{-1}O_{22}, \\ V &= O_{21}^{-1}, & N &= -O_{21}^{-1}O_{22}; \end{aligned}$$

(b) if O_{12} is invertible, then

$$\text{LFT}(O_y, Q) = (V + QN)^{-1}(U + QM),$$

with

$$\begin{aligned} U &= O_{12}^{-1}O_{11}, & M &= O_{21} - O_{22}O_{12}^{-1}O_{11}, \\ V &= O_{12}^{-1}, & N &= -O_{22}O_{12}^{-1}. \end{aligned}$$

D General Bézout Identity

For the purposes of Sections 6 and 7, the following lemma describes a generalized version of the Bézout identity.

Lemma 17 Suppose (N, M) and (\hat{N}, \hat{M}) specify the right and left coprime factorizations in (28). Then for any given $U, V, \hat{U}, \hat{V} \in \mathcal{RH}_\infty$, satisfying the Bézout identities (29), there exist their modified versions $U', V', \hat{U}', \hat{V}' \in \mathcal{RH}_\infty$ which satisfy the general Bézout identity:

$$\begin{bmatrix} \hat{V}' & -\hat{U}' \\ -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} M & U' \\ N & V' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (\text{D1})$$

PROOF. Consider the following modifications of U, V, \hat{U}, \hat{V} :

$$U' := U, \quad \hat{U}' := \hat{U} - \Upsilon \hat{M}, \quad (\text{D2})$$

$$V' := V, \quad \hat{V}' := \hat{V} - \Upsilon \hat{N}, \quad (\text{D3})$$

or

$$U' := U + M\Upsilon, \quad \hat{U}' := \hat{U}, \quad (\text{D4})$$

$$V' := V + N\Upsilon, \quad \hat{V}' := \hat{V}, \quad (\text{D5})$$

where

$$\Upsilon := \hat{U}V - \hat{V}U. \quad (\text{D6})$$

By using (29) and the relation $\hat{M}N = \hat{N}M$ (following from (28)), it can be shown that the new factors $U', V', \hat{U}', \hat{V}'$, defined either by (D2) and (D3) or by (D4) and (D5) with the same Υ from (D6), belong to \mathcal{RH}_∞ and satisfy (32). ■

Note that the transfer function Υ in (D6) vanishes if and only if $\hat{U}V = \hat{V}U$ holds, in which case, (C1) reduces to (38).

E (J, J) -Unitary Constraint and Youla Parameter

In what follows, we show how the relation (44), given in Section 7, imposes constraints on the state-space realization of the Youla parameter. The conditions for the Youla parameter in (39) can be reformulated as

$$\left(\begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} M \\ N \end{bmatrix} Q \right) \sim J_T \left(\begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} M \\ N \end{bmatrix} Q \right) = 0, \quad (\text{E1})$$

where $J_T := \text{diag}(J_\mu, -J_\mu)$. Now, suppose the state-space realization $Q = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array} \right] \in \mathcal{RH}_\infty$, and the conditions of Lemma 4 for stabilizability and detectability of \mathcal{P}_{22} are satisfied. Here, $A_Q \in \mathbb{C}^{k \times k}$, and B_Q, C_Q and D_Q are complex matrices of appropriate dimensions. Then

$$0 = \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right] \sim J_T \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right] = \left[\begin{array}{c|c} A_T & 0 \\ \hline -C_T^* J_T C_T & -A_T^* \end{array} \middle| \begin{array}{c} B_T \\ -C_T^* J_T D_T \end{array} \right]. \quad (\text{E2})$$

Here,

$$A_T := \begin{bmatrix} A_Q & 0 & 0 \\ B_2 C_Q & A & 0 \\ 0 & 0 & A \end{bmatrix}, B_T := \begin{bmatrix} B_Q \\ B_2 D_Q \\ -L \end{bmatrix}, D_T := \begin{bmatrix} D_Q \\ D_{22} D_Q + I \end{bmatrix} \quad (\text{E3})$$

$$C_T := \begin{bmatrix} C_Q & F & F \\ D_{22} C_Q & C_{22} + D_{22} F & C_{22} + D_{22} F \end{bmatrix} \quad (\text{E4})$$

and $\tilde{A} := A + B_2 F$, and use is made of the realizations of M, N, U and V given in (30) and standard addition and multiplication operations on the transfer matrices. For Hurwitz matrices A_Q and \tilde{A} , the block lower triangular matrix A_T is Hurwitz. Consequently, there exists a Hermitian matrix $\Theta_T \in \mathbb{C}^{(n+n_0) \times (n+n_0)}$ such that

$$\Theta_T A_T + A_T^* \Theta_T + C_T^* J_T C_T = 0. \quad (\text{E5})$$

As a result, (E2) is equivalent to

$$0 = \left[\begin{array}{c|c} A_T & 0 \\ \hline 0 & -A_T^* \end{array} \middle| \begin{array}{c} B_T \\ -(\Theta_T B_T + C_T^* J_T D_T) \end{array} \right], \quad (\text{E6})$$

which is derived by applying a similarity transformation $\begin{bmatrix} I & 0 \\ -\Theta_T & I \end{bmatrix}$ to the transfer function on the right-hand side of (E2). Then we can apply an additive decomposition on the right-hand side of (E6) which implies

$$0 = \left[\begin{array}{c|c} A_T & B_T \\ \hline B_T^* \Theta_T + D_T^* J_T C_T & 0 \end{array} \right] + \left[\begin{array}{c|c} A_T & B_T \\ \hline B_T^* \Theta_T + D_T^* J_T C_T & 0 \end{array} \right] \sim + D_T^* J_T D_T. \quad (\text{E7})$$

Therefore (E6) is satisfied if

$$D_T^* J_T D_T = 0, \quad (\text{E8})$$

$$B_T^* \Theta_T + D_T^* J_T C_T = 0. \quad (\text{E9})$$

In fact, for a given matrix $D_Q \in \mathbb{C}^{\mu \times \mu}$, the stabilizing problem can be solved by finding a Hurwitz matrix $A_Q \in \mathbb{C}^{n_0 \times n_0}$, and arbitrary matrices $B_Q \in \mathbb{C}^{n_0 \times \mu}$, $C_Q \in \mathbb{C}^{\mu \times n_0}$ such that the conditions in (E5), (E8) and (E9) are satisfied. These conditions, except for (E8), resemble the necessary and sufficient constraints for a minimal state-space realization $\begin{bmatrix} A_T & B_T \\ C_T & D_T \end{bmatrix}$ to be PR; see [17] for more details.

F Computation of $\text{Proj}_{\mathcal{S}}(\partial_Q E)$:

For the numerical scheme provided in Section 9, we define the projection operator as

$$\text{Proj}_{\mathcal{S}}(\partial_Q E) := \arg \min \left\{ \frac{1}{2} \|\partial_Q E - X\|_2 : X \in \mathcal{S} \right\}, \quad (\text{F1})$$

where \mathcal{S} is defined in (60). The solution to this problem is provided in the following lemma. In this lemma the projection operator $\text{Proj}_{\mathcal{H}_2} : \mathcal{L}_2 \rightarrow \mathcal{H}_2$ is defined as

$$\text{Proj}_{\mathcal{H}_2}(X) := \mathcal{F}(\tilde{x}(t)), \quad (\text{F2})$$

where $\tilde{x}(t) := \begin{cases} x(t) & t \geq 0 \\ 0 & \text{o.w.} \end{cases}$ is the causal part of the inverse Fourier transform $x(t) := \mathcal{F}^{-1}(X)$, and \mathcal{F} denotes the Fourier transform.

Lemma 18 *Consider the projection problem defined in (F1). The solution to this problem can be formulated as*

$$\text{Proj}_{\mathcal{S}}(\partial_Q E) = \text{Proj}_{\mathcal{H}_2} P, \quad (\text{F3})$$

where $P := \partial_Q E + (\Lambda + \Pi Q)\Upsilon$ and $\Upsilon = \Upsilon^*$ is chosen such that

$$(\text{Proj}_{\mathcal{H}_2} P)^*(\Lambda + \Pi Q) + (\Lambda^* + Q^* \Pi) \text{Proj}_{\mathcal{H}_2} P = 0. \quad (\text{F4})$$

PROOF. In view of the sesquilinear constraint in (59), the optimization problem in (F1) can be formulated by applying the Lagrange method as follows:

$$\mathcal{E} := \|\partial_Q E - X\|_2^2 - \langle \Upsilon, X^*(\Lambda + \Pi Q) + (\Lambda^* + Q^* \Pi)X \rangle \rightarrow \min, \quad (\text{F5})$$

over $X \in \mathcal{H}_2$ and Υ , where $\Upsilon^* = \Upsilon$ is the Lagrange multiplier. The Fréchet derivatives of the augmented cost functional \mathcal{E} defined in (F5) are given by

$$\partial_X \mathcal{E} = \text{Proj}_{\mathcal{H}_2}(X - \partial_Q E - (\Lambda + \Pi Q)\Upsilon), \quad (\text{F6})$$

$$\partial_{\Upsilon} \mathcal{E} = X^*(\Lambda + \Pi Q) + (\Lambda^* + Q^* \Pi)X. \quad (\text{F7})$$

Since the problem (F1) is a convex optimization problem (this can be shown by decomposing X into real and imaginary parts), the necessary conditions of optimality, that is, $\partial_X \mathcal{E} = 0$ and $\partial_{\Upsilon} \mathcal{E} = 0$, coincide with the sufficient conditions of optimality. Then, in view of (F6) and (F7), the optimal solution can be given by (F3), where P satisfies the constraint given in (F4). ■

Remark: As an alternative, the optimization problem (F1) can also be solved in the time domain. Then, in view of the interpolation constraints for the (J, J) -unitary systems, the solution can be transformed to the frequency domain.